Yadanabon University Research Journal, 2019, Vol-10, No.1

Existence of Solutions of Navier-Stokes Equations

Thit Thit Win*

Abstract

In this paper, some function spaces, definitions and lemmas are presented first. Next, the variational formulation and the initial boundary value problem of the Navier-Stokes equations are discussed. Finally, the existence of solutions of Navier-Stokes equations are determined.

1. Preliminaries

1.1 Some Function Spaces

Let Ω be an open bounded set in $\mathbb{R}^n$. We assume

$$\mathcal{V} = \left\{ v \in \mathcal{D}(\Omega) \mid \operatorname{div} v = 0 \right\}, \tag{1}$$

$$\begin{aligned} V &= \text{the closure of } \mathcal{V} \text{ in } H_0^1(\Omega), \\ &= \left\{ v \in H_0^1(\Omega) \mid \operatorname{div} v = 0 \right\} \end{aligned} \tag{2}$$

$$\begin{aligned} H &= \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega), \\ &= \left\{ v \in L^2(\Omega) \mid \operatorname{div} v = 0, \Gamma_\nu v = v \cdot \nu \mid \Gamma = 0 \right\}. \end{aligned} \tag{3}$$

The space H is equipped with the scalar product $(\cdot,\cdot)$ induced by $L^2(\Omega)$; the space V is a Hilbert space with the scalar product

$$\left((u,v)\right) = \sum_{i=1}^{n} \left(D_i u, D_i v\right), \tag{4}$$

since Ω is bounded. The space V is contained in H and it is dense in H. The injection is continuous. Let H' and V' denote the dual spaces of H and V, and let i denote the injection mapping from V into H. The adjoint operator i' is linear and continuous from H' into V', and i' is one to one since $i(V) = V$ is dense in H and $i'(H')$ is dense in V' since i is one to one; therefore H' can be identified with a dense subspace of V'.

Moreover, by the Riesz representation theorem, we can identify H and H', and we arrive at the inclusions

$$V \subset H \equiv H' \subset V', \tag{5}$$

where each space is dense in the following one and the injections are continuous.

As a consequence of the previous identifications, the scalar product in H of $f \in H$ and $u \in V$ is the same as the scalar product of f and u in the duality between V' and V such that

$$\langle f, u \rangle = (f, u), \ \forall f \in H, \ \forall u \in V. \tag{6}$$

For each u in V, the form

$$v \in V \to \left((u,v)\right) \in \mathbb{R} \tag{7}$$

is linear and continuous on V; therefore there exists an element of V' which we denote by Au such that

$$\langle Au, v \rangle = \left((u,v)\right), \forall v \in V, \tag{8}$$

where the mapping $u \to Au$ is linear and continuous and is an isomorphism from V onto V'.

* Associate Professor, Dr, Department of Mathematics, Yadanabon University

Yadanabon University Research Journal, 2019, Vol-10, No.1

1.2 Definition

Let a, b be two extended real numbers, $-\infty \le a < b \le \infty$, and let X be a Banach space. For given α, $1 \le \alpha < +\infty$, $L^{\alpha}(a,b;X)$ denotes the space of L^{α}-integrable functions from $[a,b]$ into X, which is a Banach space with the norm

$$\left\{ \int_a^b \|f(t)\|_X^{\alpha} \, dt \right\}^{\frac{1}{\alpha}}.$$

1.3 Definition

The space $L^{\infty}(a,b;X)$ is the space of essentially bounded functions from $[a,b]$ into X and is equipped with the Banach norm

$$\operatorname{Ess}\sup_{[a,b]} \|f(t)\|_X .$$

1.4 Definition

The space $C([a,b];X)$ is the space of continuous functions from $[a,b]$ into X and if $-\infty < a < b < \infty$ is equipped with the Banach norm

$$\sup_{t\in[a,b]} \|f(t)\|_X .$$

1.5 Lemma

Let X be a given Banach space with dual X' and let u and g be two functions belonging to $L^1(a,b;X)$. Then, the following three conditions are equivalent

(i) u is a.e equal to a primitive function of g,

$$u(t) = \xi + \int_0^t g(s)ds, \ \xi \in X, \text{ a.e, } t \in [a,b]; \qquad (9)$$

(ii) For each test function $\phi \in \mathcal{D}((a,b))$,

$$\int_a^b u(t)\phi'(t)dt = -\int_a^b g(t)\phi(t)dt \left(\phi' = \frac{d\phi}{dt} \right); \qquad (10)$$

(iii) For each $\eta \in X'$,

$$\frac{d}{dt}\langle u,\eta \rangle = \langle g,\eta \rangle, \qquad (11)$$

in the scalar distribution sense, on (a,b). If (i)-(iii) are satisfied u, in particular, is a.e. equal to a continuous function from $[a,b]$ into X.

Proof: see [7].

Let X_0, X, X_1 be three Banach spaces such that $X_0 \subset X \subset X_1$, where the injections are continuous and X_i is reflexive, $i = 0,1$, the injection $X_0 \to X$ is compact.

1.6 Definition

Let $T > 0$ be a fixed finite number, and let α_0, α_1 be two finite numbers such that $\alpha_i > 1$, $i = 0,1$.

We consider the space

$$\mathcal{Y} = \mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1) \qquad (12)$$

Yadanabon University Research Journal, 2019, Vol-10, No.1

$$\mathcal{Y} = \left\{ v \in L^{\alpha_0}\left(0,T;X_0\right) \mid v' = \frac{dv}{dt} \in L^{\alpha_1}\left(0,T;X_1\right) \right\}. \qquad (13)$$

The space $\mathcal{Y}$ is provided with the norm

$$\|v\|_{\mathcal{Y}} = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v'\|_{L^{\alpha_1}(0,T;X_1)} \qquad (14)$$

which makes it a Banach space. It is evident that $\mathcal{Y} \subset L^{\alpha_0}\left(0,T;X\right)$, with a continuous injection.

Let us assume that X_0, X, X_1 are Hilbert space with

$$X_0 \subset X \subset X_1, \qquad (15)$$

the injections being continuous and the injection of X_0 into X is compact.

(16)

If v is a function from R into X_1, we denote by $\hat{v}$ its Fourier transform

$$\hat{v}(\tau) = \int_{-\infty}^{+\infty} e^{-2i\pi t\tau} v(t)dt. \qquad (17)$$

The derivative in t of order γ of v is the inverse Fourier transform of $\left(2i\pi\tau\right)^{\gamma} \hat{v}$ or

$$D_t^r v(\tau) = \left(2i\pi\tau\right)^{\gamma} \hat{v}(\tau). \qquad (18)$$

1.7 Definition

For given $\gamma > 0$, we define the space

$$\mathcal{H}^{\gamma}\left(R;X_0,X_1\right) = \left\{ v \in L^2\left(R;X_0\right) \mid D_t^{\gamma} v \in L^2\left(R;X_1\right) \right\}. \qquad (19)$$

This is a Hilbert space for the norm

$$\|v\|_{\mathcal{H}^{\gamma}(R;X_0,X_1)} = \left\{ \|v\|^2_{L^2(R;X_0)} + \left\| |\tau|^{\gamma} \hat{v} \right\|^2_{L^2(R;X_1)} \right\}^{\frac{1}{2}}.$$

We associate with any set $K \subset R$, the subspace $\mathcal{H}_K^{\gamma}$ of $\mathcal{H}^{\gamma}$ defined as the set of functions u in $\mathcal{H}^{\gamma}$ with support contained in K:

$$\mathcal{H}_K^{\gamma}\left(R;X_0,X_1\right) = \left\{ u \in \mathcal{H}^{\gamma}\left(R;X_0,X_1\right) \mid \text{support } u \subset K \right\}. \qquad (20)$$

1.8 Theorem

Let us assume that X_0, X, X_1 are Hilbert spaces which satisfy (15) and (16). Then for any bounded set K and any $\gamma > 0$, the injection of $\mathcal{H}_K^{\gamma}\left(R;X_0,X_1\right)$ into $L^2\left(R;X\right)$ is compact.

Proof: See [7].

2. NAVIER-STOKES EQUATIONS

We assume that a fluid fills a region Ω of space. If the fluid is Newtonian, then the functions ρ, p, u are governed by the momentum conservation equation (Navier-Stokes equation), by the continuity equation (mass conservation equation) and by some constitutive law connecting ρ and p:

$$\rho\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i} \right) - \mu\Delta u - \left(3\lambda + \mu\right)\Delta u + \nabla p = f, \qquad (21)$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0, \qquad (22)$$

where $\mu > 0$ is the kinematic viscosity, λ another physical parameter and $f = f(x,t)$ represents a density of force per unit volume. If the fluid is homogeneous and incompressible, then ρ is a constant independent of x and t and the equations reduce to

$$\rho\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{n} u_i \frac{\partial u}{\partial x_i}\right) - \mu\Delta u + \nabla p = f\ , \qquad (23)$$

$$\operatorname{div} u = 0 . \qquad (24)$$

Usually we take $\rho = 1$, set $\nu = \mu$ and using the differential operator $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$ arrive at

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f\ , \qquad (25)$$

with initial condition:

$$u(x,0) = u_0(x),\ x \in \Omega\ (u_0 \text{ given}) \qquad (26)$$

and boundary condition:

$$u(x,t) = \phi(x,t),\ x \in \Gamma,\ t > 0\ (\Omega \text{ bounded, } \phi \text{ given}). \qquad (27)$$

3. VARIATIONAL FORMULATION

Let Ω be a Lipschitz open bounded set in R^n and let $T > 0$ be fixed. The initial boundary value problem of the full Navier-Stokes equations is the following:

To find a vector function

$$u : \Omega \times [0,T] \to R^n$$

and a scalar function

$$p : \Omega \times [0,T] \to R ,$$

such that

$$\frac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^{n} u_i D_i u + \nabla p = f \ \text{ in } Q = \Omega \times (0,T), \qquad (28)$$

$$\operatorname{div} u = 0 \text{ in } Q, \qquad (29)$$

$$u = 0 \text{ on } \partial\Omega \times (0,T), \qquad (30)$$

$$u(x,0) = u_0(x), \text{ in } \Omega . \qquad (31)$$

As before, the functions f and u_0 are given, defined on $\Omega \times [0,T]$ and Ω respectively. Let us assume that u and p are classical solutions of (28)-(31), say $u \in C^2(\bar{Q})$, $p \in C^1(\bar{Q})$. Obviously $u \in L^2(0,T;V)$. Multiplying (28) by $v \in V$ and integrating over Ω, we have

$$\int_\Omega \frac{\partial u}{\partial t} v dx - \int_\Omega \nu(\Delta u) v dx + \int_\Omega \sum_{i=1}^{n} u_i (D_i u) v dx + \int_\Omega \nabla p v dx = \int_\Omega f v dx .$$

We define the trilinear form b by setting

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_\Omega u_i (D_i v_j) w_j dx .$$

Then we have

$$\left(\frac{\partial u}{\partial t}, v\right) + \nu\int_\Omega (\nabla u \cdot \nabla v) dx + b(u,u,v) + \int_\Omega p \operatorname{div} v\, dx = \langle f, v\rangle$$

Yadanabon University Research Journal, 2019, Vol-10, No.1

$$\frac{d}{dt}(u,v)+\nu((u,v))+b(u,u,v)=\langle f,v\rangle . \qquad (32)$$

By continuity, equation (32) will hold for each $v\in V$.

3.1 Variational Problem

For f and u_0 given with

$$f\in L^2(0,T;V') \qquad (33)$$

$$u_0\in H , \qquad (34)$$

to find u satisfying

$$u\in L^2(0,T;V) \qquad (35)$$

and

$$\frac{d}{dt}(u,v)+\nu((u,v))+b(u,u,v)=\langle f,v\rangle ,\ \forall v\in V \qquad (36)$$

$$u(0)=u_0 . \qquad (37)$$

3.2 Properties of Trilinear Form

A trilinear continuous form b has the following properties:

$$b(u,v,w)=-b(u,w,v) ,\ \forall u,v,w\in V \qquad (38)$$

$$b(u,v,v)=0 ,\ \forall u,v\in V . \qquad (39)$$

For u, v in V, we denote by $B(u,v)$ the element of V' defined by

$$\langle B(u,v),w\rangle = b(u,v,w) ,\ \forall w\in V , \qquad (40)$$

and we set

$$B(u)=B(u,u)\in V' ,\ \forall u\in V . \qquad (41)$$

3.3 Lemma

We assume that the dimension of the space is $n\le 4$ and that u belongs to $L^2(0,T;V)$. Then the function Bu defined by

$$\langle Bu(t),v\rangle = b(u(t),u(t),v) ,\ \forall v\in V ,\ \text{a.e.in } t\in[0,T] ,$$

belongs to $L^1(0,T;V')$.

Proof

For almost all t, $Bu(t)$ is an element of V', and the function $Bu : t\in[0,T]\to Bu(t)\in V'$ is measurable. Moreover, since b is trilinear continuous on V,

$$\|Bw\|_{V'}\le c\|w\|^2 ,\ \forall w\in V \qquad (42)$$

so that

$$\int_0^T \|Bu(t)\|_{V'}\,dt \le c\int_0^T \|u(t)\|^2\,dt < +\infty .$$

Therefore the function Bu is bounded in $L^1(0,T;V')$.

If u satisfies (35)-(36), then by (6), (8), and the above lemma one can write (36) as

$$\frac{d}{dt}\langle u,v\rangle = \langle f-\nu Au-Bu,v\rangle ,\ \forall v\in V .$$

Since A is linear and continuous from V into V' and $u \in L^2(V)$, therefore the function Au belongs to $L^2(0,T;V')$, the function $f - \nu Au - Bu$ belongs to $L^1(0,T;V')$. Lemma 1.5 implies then that

$$\left.\begin{aligned} & u' \in L^1(0,T;V') \\ & u' = f - \nu Au - Bu \end{aligned}\right\} \qquad (43)$$

and that u is almost everywhere equal to a continuous function from $[0,T]$ into V'. This makes (37) meaningful.

An alternate formulation of the problem (35)-(37) is:

3.4 Problem

Given f and u_0 satisfying (33)-(34), to find u satisfying

$$u \in L^2(0,T;V), u' \in L^1(0,T;V'), \qquad (44)$$

$$u' + \nu Au + Bu = f \text{ on } (0,T), \qquad (45)$$

$$u(0) = u_0. \qquad (46)$$

We showed that any solution of problem (3.1) is a solution of problem (3.4); these problems are equivalent.

The existence of solutions of these problems is ensured by the following theorem.

4. EXISTENCE RESULT

4.1 Theorem

Let the dimension n be less than or equal to 4. Let there be given f and u_0 which satisfy (33)-(34). Then there exists at least one function u which satisfies (44)-(46). Moreover,

$$u \in L^\infty(0,T;H). \qquad (47)$$

Proof

(i) We apply the Galerkin procedure. Since V is separable and $\mathcal{V}$ is dense in V, there exists a sequence $w_1, \ldots, w_m, \ldots$ of elements of $\mathcal{V}$, which is free and total in V. For each m, we define an approximate solution u_m of (36) as follows:

$$u_m = \sum_{i=1}^{m} g_{im}(t) w_i \qquad (48)$$

and

$$\left(u_m'(t), w_j\right) + \nu\left(\left(u_m(t), w_j\right)\right) + b\left(u_m(t), u_m(t), w_j\right) = \left\langle f(t), w_j \right\rangle,$$
$$t \in [0,T], j = 1, \ldots, m, \qquad (49)$$

$$u_m(0) = u_{0m}, \qquad (50)$$

where u_{0m} is the orthogonal projection in H of u_0 onto the space spanned by $w_1, \ldots, w_m$. The equation (49) forms a nonlinear differential system for the functions $g_{1m}, \ldots, g_{mm}$:

$$\sum_{i=1}^{m} (w_i, w_j) g_{im}'(t) + \nu \sum_{i=1}^{m} \left((w_i, w_j)\right) g_{im}(t)$$
$$+ \sum_{i,l=1}^{m} b(w_i, w_l, w_j) g_{im}(t) g_{lm}(t) = \left\langle f(t), w_j \right\rangle, \; j = 1, \ldots, m. \qquad (51)$$

Inverting the nonsingular matrix with elements (w_i, w_j), $1 \le i, \; j \le m$, we can write the differential equations in the usual form

Yadanabon University Research Journal, 2019, Vol-10, No.1

$$g'_{im}(t)+\sum_{j=1}^{m}\alpha_{ij}\,g_{jm}(t)+\sum_{j,k=1}^{m}\alpha_{ijk}\,g_{jm}(t)\,g_{km}(t)=\sum_{j=1}^{m}\beta_{ij}\left\langle f(t),w_j\right\rangle, \qquad (52)$$

where $\alpha_{ij},\alpha_{ijk},\beta_{ij}\in R$.

The condition (50) is equivalent to the m scalar initial conditions

$$g_{im}(0)=\text{the } i^{th} \text{ component of } u_{0m}. \qquad (53)$$

The nonlinear differential system (52) with the initial condition (53) has a maximal solution defined on some interval $[0,t_m]$. If $t_m<T$, then $|u_m(t)|$ must tend to $+\infty$ as $t\to t_m$; the priori estimates we shall prove that this does not happen and therefore $t_m=T$.

(ii) We multiply (49) by $g_{jm}(t)$ and add these equations for $j=1,\dots,m$.

Taking (39) into account, we get

$$\left(u'_m(t),u_m(t)\right)+\nu\|u_m(t)\|^2=\left\langle f(t),u_m(t)\right\rangle,$$

$$\le 2\|f(t)\|_{V'}\|u_m(t)\|. \qquad (54)$$

By using Young's inequality, we have

$$\left(u'_m(t),u_m(t)\right)+\nu\|u_m(t)\|^2\le 2\left[\frac{\nu}{2}\|u_m(t)\|^2+\frac{1}{2\nu}\|f(t)\|_{V'}^2\right].$$

And so
$$\frac{d}{dt}|u_m(t)|^2+\nu\|u_m(t)\|^2\le\frac{1}{\nu}\|f(t)\|_{V'}^2. \qquad (55)$$

Integrating (55) from 0 to s, we obtain in particular,

$$\int_0^s\frac{d}{dt}|u_m(t)|^2\,dt+\int_0^s\nu\|u_m(t)\|^2\,dt\le\int_0^s\frac{1}{\nu}\|f(t)\|_{V'}^2\,dt.$$

Then
$$|u_m(s)|^2\le|u_m(0)|^2+\frac{1}{\nu}\int_0^s\|f(t)\|_{V'}^2\,dt.$$

Hence
$$\sup_{s\in[0,T]}|u_m(s)|^2\le|u_0|^2+\frac{1}{\nu}\int_0^T\|f(t)\|_{V'}^2\,dt \qquad (56)$$

which implies that

the sequence u_m remains in a bounded set of $L^\infty(0,T;H)$. (57)

Then we integrate (55) from 0 to T to get

$$|u_m(T)|^2+\nu\int_0^T\|u_m(t)\|^2\,dt\le|u_{0m}|^2+\frac{1}{\nu}\int_0^T\|f(t)\|_{V'}^2\,dt$$

$$\le|u_0|^2+\frac{1}{\nu}\int_0^T\|f(t)\|_{V'}^2\,dt.$$

Therefore the sequence u_m remains in a bounded set of $L^2(0,T;V)$. (58)

(iii) Let $\tilde{u}_m$ denote the function from R into V which is equal to u_m on $[0,T]$ and to 0 on the complement of this interval. The Fourier transform of $\tilde{u}_m$ is denoted by $\hat{u}_m$.

We want to show that

$$\int_{-\infty}^{+\infty}|\tau|^{2\gamma}|\hat{u}_m(\tau)|^2\,d\tau\le\text{const.}, \text{ for some } \gamma>0. \qquad (59)$$

Along with (58), this will imply that

$\tilde{u}_m$ belongs to a bounded set of $\mathcal{H}^\gamma\left(R;V,H\right)$ (60)

and will enable us to apply the compactness result of Theorem 1.8.

In order to prove (59) we observe that (49) can be written

$$\left(u_m'(t), w_j\right) = \left\langle f - \nu A u_m - B u_m, w_j \right\rangle$$
$$= \left\langle f_m, w_j \right\rangle,$$

where $\tilde{f}_m = f_m$ on $[0,T]$, 0 outside this interval. By Lemma (1.5), each function u_m is after modification on a set of measure 0, continuous from $[0,T]$ into H. It is classical that since $\tilde{u}_m$ has two discontinuous at 0 and T, the distribution derivative of $\tilde{u}_m$ is given by

$$\frac{d\tilde{u}_m}{dt} = \tilde{g}_m + u_m(0)\delta_0 - u_m(T)\delta_T,$$

where δ_0, δ_T are Dirac distributions at 0 and T and

$$g_m = u_m' = \text{the derivative of } u_m \text{ on } [0,T].$$

$$\frac{d}{dt}(\tilde{u}_m, w_j) = \left\langle \tilde{f}_m, w_j \right\rangle + (u_{0m}, w_j)\delta_0 - (u_m(T), w_j)\delta_T, \ j = 1, \ldots, m. \qquad (61)$$

By the Fourier transform, we have

$$\left(\int_{-\infty}^{\infty} e^{-2\pi i t\tau} \frac{d\tilde{u}_m}{dt} dt, w_j \right) = \left\langle \int_{-\infty}^{\infty} \tilde{f}_m e^{-2\pi i t\tau} dt, w_j \right\rangle + \left(\int_{-\infty}^{\infty} u_m(0)\delta_0 e^{-2\pi i t\tau} dt, w_j \right)$$
$$- \left(\int_{-\infty}^{\infty} u_m(T)\delta_T e^{-2\pi i t\tau} dt, w_j \right).$$

Then

$$\left(e^{-2\pi i T\tau} u_m(T), w_j\right) - \left(u_m(0), w_j\right) + 2\pi i \tau \left(\int_{-\infty}^{\infty} e^{-2\pi i t\tau} \tilde{u}_m dt, w_j \right) = \left\langle \int_{-\infty}^{\infty} \tilde{f}_m e^{-2\pi i t\tau} dt, w_j \right\rangle.$$ Therefore

$$\left(e^{-2\pi i T\tau} u_m(T), w_j\right) - \left(u_m(0), w_j\right) + 2\pi i \tau \left(\hat{u}_m, w_j\right) = \left\langle \hat{f}_m, w_j \right\rangle.$$

Hence $$2\pi i \tau \left(\hat{u}_m, w_j\right) = \left\langle \hat{f}_m, w_j \right\rangle + \left(u_{0m}, w_j\right) - \left(u_m(T), w_j\right) e^{-2\pi i T\tau}, \qquad (62)$$

$\hat{u}_m$ and $\hat{f}_m$ denoting the Fourier transforms of $\tilde{u}_m$ and $\tilde{f}_m$ respectively.

We multiply (62) by $\hat{g}_{jm}(\tau)$ (= Fourier transform of $\tilde{g}_{jm}$) and add the resulting equations for $j = 1, \ldots, m$; we get

$$2i\pi\tau \left|\hat{u}_m(\tau)\right|^2 = \left\langle \hat{f}_m(\tau), \hat{u}_m(\tau) \right\rangle + \left(u_{0m}, \hat{u}_m(\tau)\right) - \left(u_m(T), \hat{u}_m(\tau)\right) e^{-2i\pi T\tau}. \qquad (63)$$

$$\int_0^T \left\|f_m(t)\right\|_{V'} dt = \int_0^T \left\|f(t) - \nu A u_m - B u_m\right\|_{V'} dt$$
$$\leq \int_0^T \left(\left\|f(t)\right\|_{V'} + \nu \left\|A u_m(t)\right\|_{V'} + \left\|B u_m(t)\right\|_{V'} \right) dt.$$

By (42), we have

$$\int_0^T \left\|f_m(t)\right\|_{V'} dt \leq \int_0^T \left(\left\|f(t)\right\|_{V'} + \nu \left\|u_m(t)\right\| + c_1 \left\|u_m(t)\right\|^2 \right) dt.$$

Since the sequence u_m remains in a bounded set of $L^2(0,T;V)$, the sequence f_m is in a bounded set of $L^1(0,T;V')$. i.e.,

$$\sup_{\tau\in R}\left\|\hat{f}_m(\tau)\right\|_{V'} \le c_2 \forall\, m.$$

By (56), we have

$$|u_m(0)| \le c_3,\ |u_m(T)| \le c_3^*.$$

From (63), we obtain

$$2\pi i|\tau||\hat{u}_m(\tau)|^2 \le \left\|\hat{f}_m(\tau)\right\|_{V'}\|\hat{u}_m(\tau)\| + |u_m(0)||\hat{u}_m(\tau)| + |u_m(T)||\hat{u}_m(\tau)|$$

$$\le c_4\|\hat{u}_m(\tau)\| \text{ where } c_4 = c_2 + c_3 + c_3^*. \tag{64}$$

For fixed $\gamma < \frac{1}{4}$, we observe that

$$|\tau|^{2\gamma} \le c_5(\gamma)\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}, \forall\, \tau \in R.$$

Thus

$$\int_{-\infty}^{+\infty}|\tau|^{2\gamma}|\hat{u}_m(\tau)|^2\, d\tau \le c_5(\gamma)\int_{-\infty}^{+\infty}\frac{1+|\tau|}{1+|\tau|^{1-2\gamma}}|\hat{u}_m(\tau)|^2\, d\tau$$

$$= c_5(\gamma)\left[\int_{-\infty}^{+\infty}\frac{|\hat{u}_m(\tau)|^2}{1+|\tau|^{1-2\gamma}}\, d\tau + \int_{-\infty}^{+\infty}\frac{|\tau||\hat{u}_m(\tau)|^2}{1+|\tau|^{1-2\gamma}}\, d\tau\right].$$

By (64), we have

$$\int_{-\infty}^{+\infty}|\tau|^{2\gamma}|\hat{u}_m(\tau)|^2\, d\tau \le c_6\int_{-\infty}^{+\infty}\frac{\|\hat{u}_m(\tau)\|}{1+|\tau|^{1-2\gamma}}\, d\tau + c_7\int_{-\infty}^{+\infty}\|\hat{u}_m(\tau)\|^2\, d\tau. \tag{65}$$

By the Parseval equality and (58), the last integral of (65) is bounded as $m \to \infty$. i.e.,

$$c_7\int_{-\infty}^{+\infty}\|u_m(t)\|^2\, dt = c_7\int_0^T\|u_m(t)\|^2\, dt \le c_8. \tag{66}$$

By the Cauchy Schwarz inequality, the first integral on the right hand side of (65) becomes

$$c_6\int_{-\infty}^{+\infty}\frac{\|\hat{u}_m(\tau)\|}{1+|\tau|^{1-2\gamma}}\, d\tau \le c_6\left(\int_{-\infty}^{+\infty}\frac{1}{\left(1+|\tau|^{1-2\gamma}\right)^2}\, d\tau\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\|\hat{u}_m(t)\|^2\, dt\right)^{\frac{1}{2}}.$$

By the Parseval equality, we have

$$c_6\int_{-\infty}^{+\infty}\frac{\|\hat{u}_m(\tau)\|}{1+|\tau|^{1-2\gamma}}\, d\tau \le c_6\left(\int_{-\infty}^{+\infty}\frac{1}{\left(1+|\tau|^{1-2\gamma}\right)^2}\, d\tau\right)^{\frac{1}{2}}\left(\int_0^T\|u_m(t)\|^2\, dt\right)^{\frac{1}{2}}$$

which is finite since $\gamma < \frac{1}{4}$ and bounded as $m \to \infty$ by (58). Therefore, (59) is proved. i.e.,

$$D_t^\gamma \tilde{u}_m \in L^2(R; H).$$

Since the sequence $\tilde{u}_m$ is bounded in $L^2(R;V)$, $\tilde{u}_m$ belongs to a bounded set of $\mathcal{H}^\gamma\left(R; V, H\right)$.

(iv) The estimates (57) and (58) enable us to assert the existence of an element $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and a sub-sequence $u_{m'}$ such that

$$\left.\begin{array}{l} u_{m'} \to u \text{ in } L^2(0,T;V) \text{ weakly, and in} \\ L^\infty(0,T;H) \text{ weak-star, as } m' \to \infty. \end{array}\right\} \tag{66}$$

By (60) and Theorem (1.8), we have

$$u_{m'} \to u \text{ in } L^2(0,T;H) \quad \text{strongly.} \tag{67}$$

The convergence results (66) and (67) enable us to pass to the limit.

Let ψ be a continuously differentiable function on $[0,T]$ with $\psi(T)=0$. We multiply (49) by $\psi(t)$ and integrate by parts, then we get

$$\int_0^T \left(u'_m(t), w_j\right)\psi(t)\,dt + \int_0^T \nu\left(u_m(t), w_j\right)\psi(t)\,dt + \int_0^T b\left(u_m(t), u_m(t), w_j\right)\psi(t)\,dt$$

$$= \int_0^T \left\langle f(t), w_j\right\rangle \psi(t)\,dt,$$

$$-\int_0^T \left(u_m(t), \psi'(t) w_j\right)dt + \nu\int_0^T \left(\left(u_m(t), w_j\psi(t)\right)\right)dt + \int_0^T b\left(u_m(t), u_m(t), w_j\psi(t)\right)dt$$

$$= (u_{om}, w_j)\psi(0) + \int_0^T \left\langle f(t), w_j\psi(t)\right\rangle dt\,. \tag{68}$$

For the nonlinear term, we have

$$\int_0^T b\left(u_m(t), u_m(t), w_j\psi(t)\right)dt = -\int_0^T b\left(u_m(t), w_j\psi(t), u_m(t)\right)dt$$

$$= -\sum_{i,j=1}^{n}\int_0^T\int_\Omega \left(u_m(t)\right)_i \left(w_j D_i \psi_j(t)\right)\left(u_m(t)\right)_j \,dxdt\,.$$

These integrals converge to

$$-\sum_{i,j=1}^{n}\int_0^T\int_\Omega \left(u(t)\right)_i \left(w_j D_i \psi_j(t)\right)\left(u(t)\right)_j \,dxdt = -\int_0^T b\left(u(t), v\psi(t), u(t)\right)dt$$

$$= \int_0^T b\left(u(t), u(t), v\psi(t)\right)dt\,.$$

In the limit we find that the equation

$$-\int_0^T \left(u(t), v\psi'(t)\right)dt + \nu\int_0^T \left(\left(u(t), v\psi(t)\right)\right)dt + \int_0^T b\left(u(t), u(t), v\psi(t)\right)dt$$

$$= (u_o, v)\psi(0) + \int_0^T \left\langle f(t), v\psi(t)\right\rangle dt\,, \tag{69}$$

holds for $v = w_1, w_2, \ldots,$ by linearity this equation holds for $v =$ any finite linear combination of the w_j and by a continuity argument (69) is still true for any $v \in V$.

By writing, in particular, (69) with $\psi = \phi \in \mathcal{D}\left((0,T)\right)$, u satisfies (36) in the distribution sense.

Finally, it remains to prove that u satisfies (37). For this, we multiply (36) by ψ, and integrate. After integrating the first term by parts, we get

$$\left[\left(u(t)\psi(t), v\right)\right]_0^T - \int_0^T \left(u(t), v\right)\psi'(t)\,dt + \nu\int_0^T \left(\left(u(t), v\psi(t)\right)\right)dt$$

$$+\int_0^T b\left(u(t), u(t), v\psi(t)\right)dt$$

$$= \int_0^T \langle f(t), v\psi(t) \rangle dt$$

$$- \int_0^T \big(u(t), v\psi'(t)\big) dt + \nu \int_0^T \big(\big(u(t), v\psi(t)\big)\big) dt + \int_0^T b\big(u(t), u(t), v\psi(t)\big) dt$$

$$= \big(u(0), v\big)\psi(0) + \int_0^T \langle f(t), v\psi(t) \rangle dt .$$

By comparison with (69),

$$\big(u(0) - u_0, v\big)\psi(0) = 0 .$$

By choosing ψ with $\psi(0) = 1$, we obtain

$$\big(u(0) - u_0, v\big) = 0, \ \forall v \in V .$$

Therefore $\qquad u(0) = u_0 .$

Acknowledgements

I would like to express my heartfelt gratitude to **Dr Maung Maung Naing**, Rector, **Dr Si Si Khin**, Pro-Rector and **Dr Tint Moe Thuzar**, Pro-Rector, Yadanabon University for their permission to carry out the research and their encouragement. And then, I would like to thank **Dr Win Kyaw**, Professor and Head of Department of Mathematics, **Dr Hla Win**, Professor, Department of Mathematics and **Dr Nan Mya Ngwe**, Professor, Department of Mathematics, Yadanabon University, for their exhortation and helpful comments on this research.

References

[1] Adams, R. A., Sobolev Spaces, Academic Press, New York, 1975.

[2] Evans, L. C., Partial Differential Equations, AMS, Providence, 1998.

[3] Girault, V. and Raviart, P-A., Finite Element Methods for Navier-Stokes Equation. Theory and Algorithms, Springer-Verlag, New York, 1986.

[4] Quarteroni, A. and Valli, A., Numerical Approximation of Partial Differential Equations, second corrected printing, Springer-Verlag, Berlin/Heidelberg, 1997.

[5] Süli, E., Finite Element Methods for Partial Differential Equations, Preprint, U Vienna, 2002.

[6] Skalak, Z. & Kucera, P., *Remark on Regularity of Weak Solutions to The Navier-Stokes Equations* Comment. Math. Univ, 2001.

[7] Teman, R., Navier-Stokes Equations Theory and Numerical Analysis, Springer-Verlag, New York, 1977.